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AN INTERNAL CHARACTERIZATION OF PARACOMPACT p-SPACES

BY

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ABSTRACT. The purpose of this paper is to characterize paracompact p-spaces in terms of spaces with refining sequences mod k. A space X has a refining sequence mod k if there exists a sequence $\{\mathcal{G}_n \mid n \in N\}$ of open covers for X such that $\bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n) = P_C^1$ is compact for each compact subset C of X and $\{\mathrm{St}(C, \mathcal{G}_n)^- \mid n \in N\}$ is a neighborhood base for P_C^1. If $P_C^1 = C$ for each compact subset C of X then X is metrizable. On the other hand if we restrict the set C to the family of finite subsets of X in the above definition then we have a characterization for strict p-spaces. Moreover, in this case, if $P_C^1 = C$ for all such sets then X is developable. Thus the concept of a refining sequence mod k is natural and it is helpful in understanding paracompact p-spaces.

1. Introduction. In 1963 Arhangel'skiĭ (see [2] and [4]) introduced the concept of a p-space. He defined a completely regular space X to be a *p-space* if there exists a sequence $\{\mathcal{H}_n \mid n \in N\}$ of open collections in the Stone-Čech compactification of X such that $\mathcal{H}_n$ covers X for each $n \in N$ and such that, for each $x \in X$, $\bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathcal{H}_n) \subseteq X$ where $\mathrm{St}(x, \mathcal{H}_n) = \bigcup\{H \mid x \in H \text{ and } H \in \mathcal{H}_n\}$. A sequence $\{\mathcal{H}_n \mid n \in N\}$ of open collections in the Stone-Čech compactification of X is called a *p-sequence* if it satisfies the above conditions. Arhangel'skiĭ characterized the class of perfect preimages of metric spaces to be the class of paracompact p-spaces which answered a question of Aleksandrov in [1].

There is another class of spaces that is related to the class of p-spaces under certain restrictions and that is the class of M-space. Morita [9] introduced the concept of an M-space in 1964. A topological space is an *M-space* if there exists a normal sequence $\{\mathcal{U}_i \mid i \in N\}$ of open covers of X satisfying the condition (M):

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$$\text{(M)}\quad \begin{cases} \text{If } \{K_i \mid i \in N\} \text{ is a sequence of nonempty subsets of } X \text{ such that} \\ K_{i+1} \subseteq K_i,\ K_i \subseteq \mathrm{St}(x_0 \mathfrak{U}_i) \text{ for each } i \in N \text{ and for some fixed point} \\ x_0 \in X, \text{ then } \bigcap_{i=1}^{\infty} K_i^- \neq \emptyset. \end{cases}$$

A sequence of covers $\{\mathfrak{U}_i \mid i \in N\}$ is a *normal sequence* if and only if $\mathfrak{U}_{i+1}$ star refines $\mathfrak{U}_i$ for each i, that is, for each $U \in \mathfrak{U}_{i+1}$ there exists $V \in \mathfrak{U}_i$ such that $\mathrm{St}(U, \mathfrak{U}_{i+1}) \subseteq V$. Morita proved in [10] that the class of quasi perfect preimages of metric spaces is precisely the class of M-spaces. A map f from a space X onto a space Y is *quasi perfect* if and only if it is closed continuous map such that $f^{-1}(y)$ is countably compact for each $y \in Y$. Thus a relationship between M-spaces and p-spaces is established.

1.1 **Theorem.** *If X is a completely regular metacompact space then X is an M-space if and only if it is a paracompact p-space.*

There is yet another approach to the concept of a paracompact p-space and it involves a property related to the concept of a development for a space. It is well known that a topological space X is *developable* if and only if there exists a sequence $\{\mathcal{G}_n \mid n \in N\}$ of open covers of X such that for each open set U in X and for each $x \in U$ there exists $n \in N$ such that $\mathrm{St}(x, \mathcal{G}_n) = \bigcup\{G \in \mathcal{G}_n \mid x \in G\} \subseteq U$. The sequence $\{\mathcal{G}_n \mid n \in N\}$ of open covers of X is called a *development* for X. If the concept of a development is strengthened by replacing the point x in the definition with a compact set C one obtains Arhangel'skiĭ's definition in [3] of a k-refining sequence, that is, a sequence of open covers $\{\mathcal{G}_n \mid n \in N\}$ is said to be a *k-refining sequence* if and only if for each compact set C contained in X and each open subset U of X containing C, there exists $n \in N$ such that $\mathrm{St}(C, \mathcal{G}_n) \subseteq U$. Arhangel'skiĭ proved in [3] that spaces with this property are metrizable, that is:

1.2 **Theorem.** *A regular topological space is metrizable if and only if it has a k-refining sequence.*

A concept is now introduced which is a natural generalization of the concept of a k-refining sequence and which will characterize paracompact p-spaces.

1.3 **Definition.**(2) A topological space X is said to have a *refining sequence*

(2) For T_2 spaces it suffices to assume that $\{\mathrm{St}(C, \mathcal{G}_n) \mid n \in N\}$ is a neighborhood base for P_C^1. Therefore in view of Lemma 2.1 we have that $\{\mathcal{G}_n \mid n \in N\}$ is a refining sequence mod k iff $\{\mathrm{St}(C, \mathcal{G}_n) \mid n \in N\}$ is a k-sequence for each compact subset C of X. The definition of k-sequence is given by E. Michael in *A quintuple quotient quest*, General Topology and Appl. 2 (1972), 91–138.

mod k if and only if there exists a sequence $\{\mathcal{G}_n \mid n \in N\}$ of open covers such that for each compact subset C of X the following hold:

(a) $\bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n) = P_C^1$ is compact.

(b) The collection $\{\mathrm{St}(C), \mathcal{G}_n)^- \mid n \in N\}$ is neighborhood base for P_C^1.

The term refining sequence mod k seems to be appropriate in light of the foregoing discussion and in view of the main result in this paper.

1.3 **Theorem.** *A Hausdorff topological space X is a paracompact p-space if and only if it has a refining sequence* mod k.

Thus it is established that the class of perfect preimages of metric spaces, the class of metacompact M-spaces, the class of paracompact p-spaces and the class of spaces with refining sequences mod k are all the same. Moreover, because of the correlation between Theorems 1.2 and 1.3 the position in which paracompact p-spaces lie is a little more apparent.

In §2 the basic properties of spaces with refining sequences mod k are studied, and the other three sections depend on this section. In addition §4 depends on the results in §3 and §5 depends on both §§3 and 4. However, if one wants to assume the results in §3 to be true, §4 can be read without reading §3. Similarly §5 can be read without reading §§3 and 4.

Throughout this paper it will be assumed that all spaces are Hausdorff. The letter N will denote the set of natural numbers and βX will denote the Stone-Čech compactification of a completely regular space X. If A is a subset of X then A^- will denote the closure of A in X and $A^-_{\beta X}$ will denote the closure of A in βX in those cases where βX is involved. A *base for the neighborhood system of the set* A is defined to be any collection of open sets with the property that if V is an open set containing A, then there exists $U \in \mathcal{U}$ with $A \subseteq U \subseteq V$. If X is a topological space with a refining sequence $\{\mathcal{G}_n \mid n \in N\}$ mod k then for any compact subset $C \subseteq X$ let $P_C^1 = \bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n)$, and by induction let $P_C^{k+1} = \bigcap_{n=1}^{\infty} \mathrm{St}(P_C^k, \mathcal{G}_n)$. In case $C = \{x\}$ for some $x \in X$ let $P_{\{x\}}^k = P_x^k$ for each $k \in N$.

An attempt has been made to use standard notation and terminology. The principal reference used in this connection is [8].

2. General properties of spaces with refining sequence mod k.

The properties developed in this section are basic to the study of spaces with refining sequences mod k. The notation and terminology introduced here will be used throughout the rest of this paper.

2.1 **Lemma.** If X *is a space with a refining sequence* $\{\mathcal{G}_n \mid n \in N\}$ mod k *then there exists a refining sequence* $\{\mathcal{H}_n \mid n \in N\}$ mod k *for* X *such that* $\mathcal{H}_{n+1}$ *refines* $\mathcal{H}_n$ *for each* $n \in N$.

Proof. Let $\mathcal{H}_1 = \mathcal{G}_1$, $\mathcal{H}_2 = \mathcal{G}_1 \cap \mathcal{G}_2$, $\mathcal{H}_3 = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3, \cdots, \mathcal{H}_n = \bigcap_{i=1}^n \mathcal{G}_i, \cdots$. We show that $\{\mathcal{H}_n \mid n \in N\}$ is the desired sequence. Clearly $\mathcal{H}_{n+1}$ refines $\mathcal{H}_n$ for each n. Let C be a compact subset of X and let $x \in P_C^1$ where $P_C^1 = \bigcap_{n=1}^\infty \mathrm{St}(C, \mathcal{G}_n)$; then $x \in \mathrm{St}(C, \mathcal{G}_n)$ for each $n \in N$. Thus $\bigcap_{i=1}^\infty \mathrm{St}(x, \mathcal{G}_i) \cap C \neq \emptyset$; for if $\bigcap_{i=1}^\infty \mathrm{St}(x, \mathcal{G}_i) \cap C = \emptyset$ then $\bigcap_{i=1}^\infty \mathrm{St}(x, \mathcal{G}_i)^- \cap C = \emptyset$ and thus there is $k \in N$ such that $\bigcap_{i=1}^k \mathrm{St}(x, \mathcal{G}_i)^- \cap C = \emptyset$. Clearly $\bigcap_{i=1}^k \mathrm{St}(x, \mathcal{G}_i) \supseteq P_x^1$ and is an open set. So there exists $n \in N$ such that $P_x^1 \subseteq \mathrm{St}(x, \mathcal{G}_n) \subseteq \bigcap_{i=1}^k \mathrm{St}(x, \mathcal{G}_i)$; but

$$\emptyset \neq \mathrm{St}(x, \mathcal{G}_n) \cap C \subseteq \bigcap_{i=1}^k \mathrm{St}(x, \mathcal{G}_i) \cap C \subseteq \bigcap_{i=1}^k \mathrm{St}(x, \mathcal{G}_i)^- \cap C,$$

a contradiction. Let $y \in \bigcap_{i=1}^k \mathrm{St}(x, \mathcal{G}_i) \cap C$. Then for each $i \leq k$ there is $G_i \in \mathcal{G}_i$ such that x and $y \in G_i$. So x and $y \in \bigcap_{i=1}^k G_i \in \mathcal{H}_k$, that is $\mathrm{St}(x, \mathcal{H}_k) \cap C \neq \emptyset$ for any $k \in N$. So $x \in \mathrm{St}(C, \mathcal{H}_k)$ for each $k \in N$ or $x \in \bigcap_{k=1}^\infty \mathrm{St}(C, \mathcal{H}_k)$ and it follows that $P_C^1 \subseteq \bigcap_{k=1}^\infty \mathrm{St}(C, \mathcal{H}_k)$. Clearly

$$\bigcap_{k=1}^\infty \mathrm{St}(C, \mathcal{H}_k) \subseteq \bigcap_{n=1}^\infty \mathrm{St}(C, \mathcal{G}_n) = P_C^1.$$

So $P_C^1 = \bigcap_{n=1}^\infty \mathrm{St}(C, \mathcal{H}_n)$. Since

$$P_C^1 \subseteq \mathrm{St}(C, \mathcal{H}_n) \subseteq \mathrm{St}(C, \mathcal{H}_n)^- \subseteq \mathrm{St}(C, \mathcal{G}_n)^-$$

it follows that $\{\mathrm{St}(C, \mathcal{H}_n)^- \mid n \in N\}$ is a neighborhood base for P_C^1 and thus that $\{\mathcal{H}_n \mid n \in N\}$ is a refining sequence mod k for X.

From now on we will assume that a refining sequence $\{\mathcal{G}_n \mid n \in N\}$ mod k has the property that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ for each $n \in N$.

2.2 **Lemma.** *If X is a space with a refining sequence* mod k *then X is a regular space.*

Proof. Let $\{\mathcal{G}_n \mid n \in N\}$ be a refining sequence mod k for X; let $x \in X$ and let B be a closed set such that $x \notin B$. Recall that $P_x^1 = \bigcap_{n=1}^\infty \mathrm{St}(x, \mathcal{G}_n)$ is compact. So there exist disjoint open sets U and V such that $x \in U$, $V \supseteq B \cap P_x^1$ since X is T_2. Also there exists $n \in N$ such that $\mathrm{St}(x, \mathcal{G}_n)^- \cap B - V = \emptyset$ since $P_x^1 \subseteq X - (B - V)$ and $X - (B - V)$ is an open set. Let G be any member of $\mathcal{G}_n$ such that $x \in G$ and let $O = G \cap U$. Now O is an open neighborhood of x and $O^- \subseteq \mathrm{St}(x, \mathcal{G}_n)^- \cap U^-$. So $O^- \cap B = \emptyset$ and thus X is regular.

2.3 **Lemma.** *Let X be a topological space with a refining sequence $\{\mathcal{G}_n \mid n \in N\}$* mod k *and let C be any compact subset of X. If $\{x_n \mid n \in N\}$ is a sequence of points in X such that $x_n \in \mathrm{St}(C, \mathcal{G}_n)$ for each $n \in N$, then $\{x_n \mid n \in N\} \cup P_C^1$ is compact. Moreover every cluster point of the sequence $\{x_n \mid n \in N\}$ is in P_C^1.*

Proof. Suppose $A \subseteq \{x_n \mid n \in N\} \cup P^1_C$ and that A has no limit points. Then $A \cap P^1_C$ is a finite set since P^1_C is compact. Also $A - P^1_C$ is closed since A has no limit points. Thus $X - (A - P^1_C)$ is an open set in X containing P^1_C and so there exists $n \in N$ such that $\mathrm{St}(C, \mathcal{G}_n) \subseteq X - (A - P^1_C)$. Thus

$$A - P^1_C \subseteq X - \mathrm{St}(C, \mathcal{G}_n) \subseteq X - \{x_k \mid k \geq n,\ k \in N\}.$$

So $A - P^1_C \subseteq \{x_k \mid k \leq n\}$ and hence A is a finite set. Therefore $\{x_k \mid k \in N\} \cup P^1_C$ is compact.

If x is a cluster point of $\{x_k \mid k \in N\}$ then $x \in [\mathrm{St}(C, \mathcal{G}_n)]^-$ for each n. Thus $x \in \bigcap_{n=1}^{\infty} [\mathrm{St}(C, \mathcal{G}_n)]^- = P^1_C$ and this completes the proof.

2.4 **Lemma.** *Let X be a topological space with a refining sequence $\{\mathcal{G}_n \mid n \in N\}$ mod k, let C be a compact subset of X and let $\{x_n \mid n \in N\}$ be a sequence in X such that $x_n \in \mathrm{St}(C, \mathcal{G}_n)$ for each $n \in N$. If $y_k \in \mathrm{St}(\{x_i \mid i \in N,\ i \geq k\}^-, \mathcal{G}_k)$ for each $k \in N$ then $(\{y_k \mid k \in N\})^-$ is compact and every cluster point of the sequence $\{y_k \mid k \in N\}$ is in $\bigcup \{P^1_x \mid x$ is a cluster point of $\{x_n \mid n \in N\}\}$.*

The proof of the above lemma is similar to the proof of Lemma 2.3 and is omitted.

2.5 **Lemma.** *Let $\{\mathcal{G}_n \mid n \in N\}$ be a refining sequence* mod k *for X and let C be a compact subset of X. Then $P^1_C = \bigcup \{P^1_z \mid z \in C\}$.*

Proof. For each n,

$$\mathrm{St}(C, \mathcal{G}_n) = \bigcup \{G \in \mathcal{G}_n \mid G \cap C \neq \emptyset\} = \bigcup \{\mathrm{St}(z, \mathcal{G}_n) \mid z \in C\} \supseteq \bigcup \{P^1_z \mid z \in C\}.$$

Thus $P^1_C \supseteq \{P^1_z \mid z \in C\}$.

Conversely, suppose $x \in P^1_C$ then for each $n \in N$ there exists $x_n \in C$ such that $x \in \mathrm{St}(x_n, \mathcal{G}_n)$. Thus $x_n \in \mathrm{St}(x, \mathcal{G}_n)$ for each $n \in N$ and thus $\{x_n \mid n \in N\}$ has a cluster point y in P^1_x. Now recall that $x_n \in C$ for each n; so $y \in C$. Moreover if $y \in P^1_x$ it follows that $x \in P^1_y$ and so $x \in \bigcup \{P^1_z \mid z \in C\}$. Thus $P^1_C = \bigcup \{P^1_z \mid z \in C\}$.

Recall that in the introduction we let $P^1_x = \bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathcal{G}_n)$, $P^2_x = \bigcap_{n=1}^{\infty} \mathrm{St}(P^1_x, \mathcal{G}_n)$ and so on. It follows from Lemma 2.5 that $P^k_x = \bigcup \{P^1_w \mid w \in P^{k-1}_x\}$ for $k \geq 2$. Thus for each $k \geq 2$ $z \in P^k_x$ if and only if there exists a finite sequence $z_0, z_1, \cdots, z_k$ such that $z_0 = x$, $z_1 \in P^1_{z_0}$, $z_2 \in P^1_{z_1}$, $\cdots, z_k \in P^1_{z_{k-1}}$ and $z_k = z$.

2.6 **Lemma.** *For each $x \in X$ let $P^{\omega}_x = \bigcap_{n=1}^{\infty} P^n_x$. Then $\{P^{\omega}_x \mid x \in X\}$ partition X.*

Proof. Suppose $x, y \in X$ and $P^{\omega}_x \cap P^{\omega}_y \neq \emptyset$. Let $z \in P^{\omega}_x \cap P^{\omega}_y$. Then $z \in P^k_x$, and $z \in P_y^m$ for some k and $m \in N$. Now $P_y^{m+1} = \bigcup\{P_w^1 \mid w \in P_y^m\}$. So $P_z^1 \subseteq P_y^{m+1}$, $P_z^2 \subseteq P_y^{m+2}, \cdots, P_z^i \subseteq P_y^{m+i}, \cdots$. If $w \in P_x^\omega$ then $w \in P_x^n$ for some $n \in N$ and there exist $w_0, \cdots, w_n$ such that $w_{i+1} \in P_{w_i}^1$ for $i = 0, 1, 2, \cdots, n-1$ where $w_0 = x$ and $w_n = w$. Also there exist $z_0, z_1, \cdots, z_k$ such that $z_{i+1} \in P_{z_i}^1$ for $i = 0, 1, 2, \cdots, n$ where $z = z_n$ and $x = z_0$. Hence it follows that $w \in P_z^{n+k} \subseteq P_y^{m+n+k} \subseteq P_y^\omega$. Thus $P_x^\omega \subseteq P_y^\omega$. By a similar argument it can be shown that $P_y^\omega \subseteq P_x^\omega$ and thus the lemma follows.

Let Γ be the first ordinal number whose cardinal number equals card $(\{P_x^\omega \mid x \in X\})$. To each $\alpha \in \Gamma$ assign an element x_α in X such that $P_{x_\alpha}^\omega \neq P_{x_\beta}^\omega$ whenever $\alpha \neq \beta$. To simplify notation we let $P_\alpha = P_{x_\alpha}^\omega$ and $P_\alpha^k = P_{x_\alpha}^k$ for $k \in N$ throughout the remaining part of this paper.

2.7 **Lemma.** *If* $\{\mathcal{G}_n \mid n \in N\}$ *is a refining sequence* mod k *for* X, *then for each* $x \in X$ *and each* $n \in N$ *there is a neighborhood* U *of* x *such that* $P_z^1 \subseteq$ St $(x, \mathcal{G}_n)$ *for each* $z \in U \cap P_x^1$.

Proof. Suppose the lemma is false; then for each neighborhood U of x there exists $z_U \in U \cap P_x^1$ such that $P_{xU}^1 - \mathrm{St}\,(x, \mathcal{G}_n) \neq \emptyset$ for some fixed $n \in N$. Let $y_U \in P_{zU}^1 - \mathrm{St}\,(x, \mathcal{G}_n)$; then $\{y_U \mid U$ is a neighborhood of $x\}$ is a net contained in $\bigcup\{P_z^1 \mid z \in P_x^1\}$. By Lemma 2.5

$$\bigcup\{P_z^1 \mid z \in P_x^1\} = P_x^2 = \bigcap_{n=1}^{\infty} \mathrm{St}(P_x^1, \mathcal{G}_n).$$

Thus the net $\{y_U \mid U$ is a neighborhood of $x\}$ has a cluster point y in P_x^2 since P_x^2 is compact and so there is a subnet $\{y_{U_\alpha} \mid \alpha \in D\}$ of $\{y_U \mid U$ is a neighborhood of $x\}$ which converges to y. Since $z_U \in U$ for each neighborhood of x it follows that $\{z_U \mid U$ is a neighborhood of $x\}$ converges to x and thus its subnet $\{z_{U_\alpha} \mid \alpha \in D\}$ converges to x.

Observe that $x \notin P_y^1$: If $x \in P_y^1$ then $y \in P_x^1$ and thus St $(x, \mathcal{G}_n)$ is a neighborhood of y. So there exists $\alpha \in D$ such that $y_{U_\alpha} \in \mathrm{St}\,(x, \mathcal{G}_n)$ contrary to the assumption that $y_{U_\alpha} \in P_{z_{U_\alpha}}^1 - \mathrm{St}\,(x, \mathcal{G}_n)$.

Recall that P_y^1 is compact; thus

$$\bigcap_{\beta \in D} (\{z_{U_\alpha} \mid \alpha \geq \beta\})^- \cap P_y^1 = \{x\} \cap P_y^1 = \emptyset$$

implies there exists $\beta_1, \beta_2, \cdots, \beta_k$ for which $\bigcap_{i=1}^k (\{z_{U_\alpha} \mid \alpha \geq \beta_i, \alpha \in D\})^- \cap P_y^1 = \emptyset$. Let $\beta \in D$ such that $\beta \geq \beta_i$ for each $i = 1, 2, \cdots, k$; then

$$\bigcap_{i=1}^{k} (\{x_{U_\alpha} \mid \alpha \geq \beta_i,\ \alpha \in D\}) \supseteq \{z_{U_\alpha} \mid \alpha \geq \beta,\ \alpha \in D\}$$

and so $(\{z_{U_\alpha} \mid \alpha \geq \beta, \alpha \in D\})^- \cap P_y^1 = \emptyset$. Hence there exists $j \in N$ such that

$\mathrm{St}(y, \mathcal{G}_j) \subseteq X - (\{z_{U_\alpha} \mid \alpha \geq \beta, \alpha \in D\})^-$, that is $\mathrm{St}(y, \mathcal{G}_j) \cap (\{z_{U_\alpha} \mid \alpha \geq \beta, \alpha \in D\})^- = \emptyset$. But $\{z_{U_\alpha} \mid \alpha \geq \beta, \alpha \in D\} \subseteq P_x^1$ by construction. So $\{z_{U_\alpha} \mid \alpha \geq \beta, \alpha \in D\}^-$ is a compact subset of P_x^1. Moreover

$$\bigcap_{k=1}^{\infty} \mathrm{St}((\{z_{U_\alpha} | \alpha \geq \beta, \alpha \in D\})^-, \mathcal{G}_k) = \bigcup \{P_w^1 | w \in (\{z_{U_\alpha} | \alpha \geq \beta, \alpha \in D\})^-\}$$

$$\supseteq \left(\bigcup \{P_{z_{U_\alpha}} | \alpha \geq \beta, \alpha \in D\}\right)^- \supseteq (\{y_{z_{U_\alpha}} | \alpha \geq \beta, \alpha \in D\})^- \supseteq \{y\}.$$

Thus $y \in \mathrm{St}((\{z_{U_\alpha} \mid \alpha \geq \beta, \alpha \in D\})^-, \mathcal{G}_j)$, a contradiction. Thus our original assumption must be false and the conclusion of the lemma follows.

2.8 **Lemma.** *If $\{\mathcal{G}_n \mid n \in N\}$ is a refining sequence* mod k *for X and if $\mathcal{G}_n^2 = \{\mathrm{St}(x, \mathcal{G}_n) \mid x \in X\}$ for each $n \in N$, then $\{\mathcal{G}_n^2 \mid n \in N\}$ is a refining sequence* mod k *and moreover $\bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n^2)^- = P_C^2$ for each compact subset C of X.*

Proof. Recall that $P_C^2 = \bigcup \{P_z^1 \mid z \in C\}$. Clearly

$$\mathrm{St}(C, \mathcal{G}_n^2) = \bigcup \{\mathrm{St}(z, \mathcal{G}_n) | \mathrm{St}(z, \mathcal{G}_n) \cap C \neq \emptyset\} = \bigcup \{\mathrm{St}(z, \mathcal{G}_n) | z \in \mathrm{St}(C, \mathcal{G}_n)\}$$

$$\supseteq \bigcup \{P_z^1 | z \in P_C^1\} = P_C^2 \quad \text{for each } n$$

and so $\bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n^2)^- \supseteq P_C^2$. We now show that $\bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n^2)^- \subseteq P_C^2$. Let $z \in \bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n^2)^-$ and let U be an arbitrary closed neighborhood of z. Since $z \in \mathrm{St}(C, \mathcal{G}_n^2)^-$ it follows that $\mathrm{St}(C, \mathcal{G}_n^2) \cap U \neq \emptyset$ for each $n \in N$; let $z_n \in \mathrm{St}(C, \mathcal{G}_n^2) \cap U$. So there exists $x_n \in X$ such that $z_n \in \mathrm{St}(x_n, \mathcal{G}_n)$ and $\mathrm{St}(x_n, \mathcal{G}_n) \cap C \neq \emptyset$. Thus $x_n \in \mathrm{St}(C, \mathcal{G}_n)$ for each n and thus $\{x_n \mid n \in N\} \cup P_C^1$ is compact by Lemma 2.3; moreover every cluster point of $\{x_n \mid n \in N\}$ is in P_C^1. Thus by Lemma 2.4 $\{z_k \mid k \in N\}$ has a cluster point w in $\bigcap_{n=1}^{\infty} \mathrm{St}(P_C^1, \mathcal{G}_n) = P_C^2$. Clearly $w \in U$ since U is closed and $U \supseteq \{z_k \mid k \in N\}$ and hence $z \in P_C^2$ since U was an arbitrary closed neighborhood of z. Thus

$$\bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n^2)^- \subseteq P_C^2 \subseteq \bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n^2)$$

and so it follows that

$$\bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n^2)^- = P_C^2 = \bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n^2).$$

It remains to show that $\{\mathrm{St}(C, \mathcal{G}_n^2)^- \mid n \in N\}$ is a base for the neighborhood system of P_C^2. Since P_C^2 is compact and X is regular it suffices to show that $\{\mathrm{St}(C, \mathcal{G}_n^2) \mid n \in N\}$ is a base for the neighborhood system of P_C^2. Let U be an open neighborhood of P_C^2. If $\mathrm{St}(C, \mathcal{G}_n^2) \not\subseteq U$ for any $n \in N$ then there exists $z_n \in \mathrm{St}(C, \mathcal{G}_n^2) - U$ for each $n \in N$. It can be shown in precisely the same way as was done above that the sequence $\{z_n \mid n \in N\}$ has a cluster point in P_C^2 contrary to the assumption that $z_n \notin U$ for any $n \in N$. So there must exist $n \in N$ such that $\mathrm{St}(C, \mathcal{G}_n) \subseteq U$ and this completes the proof of the lemma.

Finally we prove a result which is fundamental in proof that a space with a refining sequence mod k is normal and in the proof that it is paracompact.

2.9 Lemma. *If X is a space with a refining sequence $\{\mathcal{G}_n \mid n \in N\}$ mod k and if $\mathfrak{D}$ is a closed discrete collection in X, then for each $x \in X$ and $k \in N$ there exist $n(x, k) \in N$ and a finite subset $\mathfrak{D}(x, k)$ of $\mathfrak{D}$ such that* $\mathrm{St}(P_x^k, \mathcal{G}_{n(x,k)}) \cap \mathrm{St}(E_x^k, \mathcal{G}_{n(x,k)}) = \varnothing$ *where* $E_x^k = \bigcup \{D \mid D \in \mathfrak{D} - \mathfrak{D}(x, k)\}$.

Proof. Suppose the lemma is false. Then for some $x \in X$ and some $k \in N$ there exists a sequence $\{D_n \mid n \in N\} \subseteq \mathfrak{D}$ such that $D_j \neq D_i$ for $i \neq j$ and $\mathrm{St}(P_x^k, \mathcal{G}_j) \cap \mathrm{St}(D_j, \mathcal{G}_j) \neq \varnothing$. Let $x_j \in \mathrm{St}(P_x^k, \mathcal{G}_j) \cap \mathrm{St}(D_j, \mathcal{G}_j)$; then by Lemma 2.3 $C = \{x_j \mid j \in N\} \cup P_x^{k+1}$ is compact where $P_x^{k+1} = \bigcap_{n=1}^{\infty} \mathrm{St}(P_x^k, \mathcal{G}_j)$. Moreover $\mathrm{St}(C, \mathcal{G}_j) \cap D_j \neq \varnothing$ for any $j \in N$. Let $z_j \in D_j \cap \mathrm{St}(C, \mathcal{G}_j)$; then by Lemma 2.3 $\{z_j \mid j \in N\} \cup P_C^1$ is compact where $P_C^1 = \bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{G}_n)$. Hence $\{z_j \mid j \in N\}$ has a cluster point; but $z_j \in D_j$, $z_j \notin D_i$ for $i \neq j$ and $\{D_j \mid j \in N\}$ is discrete, a contradiction. Thus the conclusion of the lemma follows.

3. A space with a refining sequence of covers mod k is subparacompact. In [6] Burke gives a number of equivalent conditions each of which characterizes subparacompactness. In the present situation only the characterization in terms of σ-discrete refinements is needed.

3.1 Definition. A space X is subparacompact if and only if every open cover for X has a σ-discrete closed refinement.

The following proposition gives a condition which is fairly obviously equivalent to Definition 2.1 and the proof is omitted.

3.2 Proposition. *Let X be a regular topological space. For any open cover $\mathfrak{U}$ of X let $\mathfrak{U}^*$ be the collection of all countable unions of members of $\mathfrak{U}$. Then X is subparacompact if and only if $\mathfrak{U}^*$ has a closed σ-discrete refinement for an arbitrary open cover $\mathfrak{U}$ of X.*

3.3 Theorem. *Every T_2 topological space with a refining sequence mod k is subparacompact.*

Proof. Let X be a topological space with a refining sequence $\{\mathcal{G}_n \mid n \in N\}$ mod k. For each $n, k \in N$ and each $x \in X$ let $V_n^k(x) = X - \bigcup \{\mathrm{St}(y, \mathcal{G}_k) \mid x \notin \mathrm{St}(y, \mathcal{G}_n)\}$ and let $V_n(x) = \bigcup_{k=1}^{\infty} V_n^k(x)$. Observe:

(1) $V_n(x) \subseteq \mathrm{St}(x, \mathcal{G}_n)$ for each $x \in X$ and each $n \in N$.

Proof. Let $z \in V_n(x)$ then there exists $k_0 \in N$ such that $z \in V_n^{k_0}(x)$ and thus $z \notin \mathrm{St}(y, \mathcal{G}_{k_0})$ whenever $x \notin \mathrm{St}(y, \mathcal{G}_n)$. Clearly $z \in \mathrm{St}(z, \mathcal{G}_{k_0})$; thus $x \in \mathrm{St}(z, \mathcal{G}_n)$ and $z \in \mathrm{St}(x, \mathcal{G}_n)$. Hence $V_n(x) \subseteq \mathrm{St}(x, \mathcal{G}_n)$.

(2) The set $V_n(x)$ is an open neighborhood of x for each n.

Proof. Clearly $x \in V_n(x)$ for each $n \in N$. Let $z \in V_n(x)$; then there exists $k_0 \in N$ such that $z \in V_n^{k_0}(x)$. By Lemma 2.7 there exists a neighborhood U of z such that if $w \in U \cap P_z^1$, then $P_w^1 \subseteq \mathrm{St}(z, \mathcal{G}_{k_0})$. Since X is regular we may choose U to be a closed neighborhood of z. Suppose

$$z \in \left(\bigcup \{\mathrm{St}(y, \mathcal{G}_k) | x \notin \mathrm{St}(y, \mathcal{G}_n)\}\right)^-$$

for each $k \in N$. Then we may choose a point

$$z_k \in U \cap \mathrm{St}(z, \mathcal{G}_k) \cap \left(\bigcup \{\mathrm{St}(y, \mathcal{G}_k) | x \notin \mathrm{St}(y, \mathcal{G}_n)\}\right)$$

for each $k \in N$. Thus there is $y_k \in X$ such that $z_k \in \mathrm{St}(y_k, \mathcal{G}_k)$ and $x \notin \mathrm{St}(y_k, \mathcal{G}_n)$ for each $k \in N$. Now $z_k \in \mathrm{St}(z, \mathcal{G}_k)$ and $y_k \in \mathrm{St}([\{z_i \mid i \in N,\ i \geq k\}]^-, \mathcal{G}_k)$ for each $k \in N$. Hence by the first part of Lemma 2.4 $(\{y_k \mid k \in N\})^-$ is compact. Let y be a cluster point of $\{y_k \mid k \in N\}$ then by the second part of Lemma 2.4 y is in P_w^1 where w is a cluster point of $\{z_i \mid i \in N\}$. By Lemma 2.3 $w \in P_z^1$ and because $z_i \in U$ for each i and U is closed, $w \in U$. Hence it follows that $y \in P_w^1 \subseteq \mathrm{St}(z, \mathcal{G}_{k_0})$ by the choice of U. Thus there is $k \in N$ such that $y_k \in \mathrm{St}(z, \mathcal{G}_{k_0})$ since y is a cluster point of $\{y_k \mid k \in N\}$. Recall that $x \notin \mathrm{St}(y_k, \mathcal{G}_n)$ and that $z \in V_n^{k_0}(x)$; thus $z \notin \mathrm{St}(y_k, \mathcal{G}_{k_0})$ and hence $y_k \notin \mathrm{St}(z, \mathcal{G}_{k_0})$, a contradiction. Hence there exists $k \in N$ such that $z \notin (\bigcup \{\mathrm{St}(y, \mathcal{G}_k) | x \notin \mathrm{St}(y, \mathcal{G}_n)\})^-$ and so

$$z \in X - \left(\bigcup \{\mathrm{St}(y, \mathcal{G}_k) | x \notin \mathrm{St}(y, \mathcal{G}_n)\}\right)^- \subseteq V_n^k(x) \subseteq V_n(x).$$

Thus it follows that $V_n(x)$ is an open set.

Let $<$ be a well order relation on X and let $V_n^k(x)^* = V_n^k(x) - \bigcup\{V_n(y) \mid y < x\}$. If $\mathcal{V}_n^k = \{V_n^k(x)^* \mid x \in X\}$ then it is clear that $\mathcal{V}_n^k$ is a disjoint collection of closed sets such that $\bigcup_{k=1}^{\infty} \mathcal{V}_n^k$ covers X for each $n \in N$. We show that $\mathcal{V}_n^k$ is closure preserving and hence discrete for each k and $n \in N$. Let $A \subseteq X$ and suppose $x \in (\bigcup\{V_n^k(y)^* \mid y \in A\})^-$. Since P_x^1 is compact and $\mathrm{St}(x, \mathcal{G}_k) \supseteq P_x^1$ there are $G_1, \cdots, G_p \in \mathcal{G}_k$ such that $x \in G_i$ for $i = 1, 2, \cdots, p$ and $\bigcup_{i=1}^p G_i \supseteq P_x^1$. In addition there is $j \geq k$ such that $\mathrm{St}(x, \mathcal{G}_j) \subseteq \bigcup_{i=1}^p G_i$. Let w be the first element in X such that $x \in V_n(w)$. Then for each $y \in A$, $y < w$ there exists $z(y) \in X$ such that $x \in \mathrm{St}(z(y), \mathcal{G}_j)$ and $y \notin \mathrm{St}(z(y), \mathcal{G}_n)$. In particular note that $z(y) \in \mathrm{St}(x, \mathcal{G}_j)$ and thus $z(y) \in G_i$ for some $i = 1, 2, \cdots, p$ and for each $y \in A$ where $y < w$. Let $W_i = (\bigcup\{V_n^k(y)^* \mid y \in A,\ y < w \text{ and } z(y) \in G_i\})^-$ for $i = 1, 2, \cdots, p$. Let $W_0 = (\bigcup\{V_n^k(y)^* \mid y \in A \text{ and } y > w\})^-$. Then

$$\left(\bigcup\{V_n^k(y)^*|y \in A\}\right)^- \subseteq \left(\bigcup_{i=0}^{p} W_i\right) \cup V_n^k(w)^*.$$

Thus $x \in W_i$ for some $i = 0, 1, 2, \cdots, p$ or $x \in V_n^k(w)^*$ and $w \in A$. Clearly $x \notin W_0$ since x is in the open set $V_n(w)$ and $V_n(w) \cap (\bigcup\{V_n^k(y)^*| y > w\}) = \emptyset$. If $x \in W_i$ for some $i = 1, 2, \cdots, p$, then $G_i \cap (\bigcup\{V_n^k(y)^*| y \in A, y < w,$ and $z(y) \in G_i\}) \neq \emptyset$ since $x \in G_i$. Let $z \in G_i \cap (\bigcup\{V_n^k(y)^*| y \in A, y < w$ and $z(y) \in G_i\})$. Then there is $y \in A$, $y < w$ such that $z \in V_n^k(y)^*$ and $z(y) \in G_i$. Moreover $z \in G_i$; so $z \in \mathrm{St}\,(z(y), \mathcal{G}_k)$; but recall $y \notin \mathrm{St}\,(z(y), \mathcal{G}_n)$. Thus $z \notin V_n^k(y)^*$, a contradiciton. So $x \notin W_i$ for $i = 0, 1, 2, \cdots, p$. Hence $x \in V_n^k(w)^*$ and $w \in A$; therefore $(\bigcup\{V_n^k(y)^*| y \in A\})^- = \bigcup\{V_n^k(y)^*| y \in A\}$ and hence $\mathcal{V}_n^k$ is discrete.

Finally let $\mathfrak{U}$ be any open cover for X and let $\mathfrak{U}^*$ be the set of all countable unions of members of $\mathfrak{U}$. Then for each $x \in X$ there is $U^* \in \mathfrak{U}^*$ such that $P_x^2 \subseteq \mathfrak{U}^*$ since P_x^2 is compact. By Lemma 2.8 there is $n \in N$ such that $\mathrm{St}\,(x, \mathcal{G}_n^2) = \bigcup\{\mathrm{St}\,(y, \mathcal{G}_n) \mid x \in \mathrm{St}\,(y, \mathcal{G}_n)\} \subseteq U^*$. Now there is $k \in N$ and $y \in X$ such that $x \in V_n^k(y)^*$; but $V_n^k(y)^* \subseteq \mathrm{St}\,(y, \mathcal{G}_n)$ and $\mathrm{St}\,(y, \mathcal{G}_n) \subseteq U^*$ since $x \in \mathrm{St}\,(y, \mathcal{G}_n)$. Thus $x \in V_n^k(y)^* \subseteq U^*$. It follows that $\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \mathfrak{U}_n^k(\mathfrak{U})$ is a closed σ-discrete refinement of $\mathfrak{U}^*$ where $\mathfrak{U}_n^k(\mathfrak{U}) = \{V_n^k(y)^* \in \mathcal{V}_n^k|\ V_n^k(y)^* \subseteq U^*$ for some $U^* \in \mathfrak{U}^*$ for each k and $n \in N\}$. Hence by Proposition 3.2 X is subparacompact.

4. A space with a refining sequence mod k is normal. In the proof that a space with a refining sequence mod k is normal the fact that such a space is subparacompact is needed and in addition the proof relies heavily on Lemma 2.9 and the following lemma.

4.1 Lemma. *Let A and B be a closed disjoint subsets of X. Then there exists an open cover $\mathfrak{U}$ for X such that if $U \in \mathfrak{U}$ and $U \cap A \neq \emptyset$ then $U^- \cap B$ or if $U \cap B \neq \emptyset$ then $U^- \cap A = \emptyset$.*

Proof. For each $x \in A$ there exist open sets U_x and $U(B)$ such that $x \in U_x$, $B \subseteq U(B)$ and $U_x \cap U(B) = \emptyset$. Similarly for each $x \in B$ there exist open sets U_x and $U(A)$ such that $x \in U_x$, $A \subseteq U(A)$ and $U_x \cap U(A) = \emptyset$. Let

$$\mathfrak{U} = \{U_x | x \in A \cup B\} \cup \{X - A \cap X - B\}.$$

If $U_x \in \mathfrak{U}$ such that $U_x \cap A \neq \emptyset$ then $x \in A$ and clearly $U_x^- \cap B = \emptyset$. Similarly if $U_x \cap B \neq \emptyset$ then $x \in B$ and $U_x^- \cap A = \emptyset$. Thus $\mathfrak{U}$ is the desired open cover.

4.2 Theorem. *A T_2 space X with a refining sequence* mod k *is normal.*

Proof. Let A and B be disjoint closed subsets of X and let $\mathfrak{U}$ be an open cover for X satisfying Lemma 4.1 relative to A and B. Since X is subparacompact there exists a σ-discrete closed refinement $\mathfrak{D} = \bigcup_{m=1}^{\infty} \mathfrak{D}_n$ for $\mathfrak{U}$.

Let n be a fixed natural number and let $A_n = A \cap (\bigcup \{D \mid D \in D_n\})$. For each $D \in \mathfrak{D}_n$ let $U(D) \in \mathfrak{U}$ such that $D \subseteq U(D)$. By Lemmas 2.8 and 2.9 for each $x \in X$ there exist n_x and a finite subset $\mathfrak{D}(n, x)$ of $\mathfrak{D}_n$ such that $\operatorname{St}(x, \mathcal{G}^2_{n_x}) \cap \operatorname{St}(E^n_x, \mathcal{G}^2_{n_x}) = \emptyset$ where $E^n_x = \bigcup \{D \mid D \in \mathfrak{D}_n - \mathfrak{D}(n, x)\}$. If $x \in B$ let

$$W_x = \operatorname{St}(x, \mathcal{G}_{n_x}) - \left(\bigcup \{U(D) \mid D \in \mathfrak{D}(n, x) \text{ and } D \cap A \neq \emptyset\}\right)^-.$$

By Lemma 4.1 $U(D)^- \cap B = \emptyset$ for any $D \in \mathfrak{D}_n$ such that $D \cap A \neq \emptyset$. Thus $(\bigcup \{U(D) \mid D \in \mathfrak{D}(n, x) \text{ and } D \cap A \neq \emptyset\})^- \cap B = \emptyset$ since $\mathfrak{D}(n, x)$ is a finite set and so W_x is an open neighborhood of x. Let $W_k = \bigcup \{W_x \mid x \in B \text{ and } n_x = k\}$. Clearly W_k is an open set and $\bigcup_{k=1}^{\infty} W_k \supseteq B$. Moreover $W_k^- \cap A_n = \emptyset$ for any $k \in N$. For suppose $z \in W_k^- \cap A_n$; then there exists $D_z \in \mathfrak{D}_n$ such that $z \in D_z$. Let $G \in \mathcal{G}_k$ such that $z \in G$; then $G \cap U(D_z) \cap W_k \neq \emptyset$ since $z \in W_k^-$. Let $y \in G \cap U(D_z) \cap W_k$; then there exists $x \in B$ such that $n_x = k$ and $y \in W_x \subseteq \operatorname{St}(x, \mathcal{G}_{n_x})$. Hence it follows that $z \in \operatorname{St}(x, \mathcal{G}^2_{n_x})$. So $D_z \in \mathfrak{D}(n, x)$ since $\operatorname{St}(x, \mathcal{G}^2_{n_x}) \cap \operatorname{St}(D, \mathcal{G}^2_{n_x}) = \emptyset$ if $D \in \mathfrak{D}_n - \mathfrak{D}(n, x)$ and hence $W_x \subseteq X - U(D_z)$. But $y \in U(D_z) \cap W_x$, a contradiction. Thus it follows that for any $k \in N$, $W_k^- \cap A_n = \emptyset$. Now if $x \in A_n$ let $U_x = \operatorname{St}(x, \mathcal{G}_{n_x}) \cap U(D_x)$ where $D_x \in \mathfrak{D}_n$ such that $x \in D_x$. Let $U(m, n) = \bigcup \{U_x \mid x \in A_n \text{ and } n_x = m\}$; clearly $U(m, n)$ is an open set and $\bigcup_{m=1}^{\infty} U(m, n) \supseteq A_n$. Moreover $U(m, n)^- \cap B = \emptyset$ for any $m \in N$. For suppose $z \in U(m, n)^- \cap B$; then there exists $G \in \mathcal{G}_m$ such that $z \in G$ and $G \cap U(m, n) \neq \emptyset$; let $y \in G \cap U(m, n)$. Then there exists $x \in A_n$ such that $n_x = m$ and $y \in U_x \subseteq \operatorname{St}(x, \mathcal{G}_{n_x})$. Thus it follows that $z \in \operatorname{St}(x, \mathcal{G}^2_{n_x})$. Note that if

$$s \in U(m, n) - \left(\bigcup \{U(D) \mid D \cap A \neq \emptyset \text{ and } D \in \mathfrak{D}(n, x)\}\right)^-$$

then there exist $w \in A_n$ and $D_w \in \mathfrak{D}_n$ such that $n_w = m$, $w \in D_w \cap A$ and $s \in U_w = \operatorname{St}(w, \mathcal{G}_{n_w}) \cap U(D_w)$. Thus since $s \in U(D_w)$ and $s \notin \bigcup \{U(D) \mid D \cap A \neq \emptyset \text{ and } D \in \mathfrak{D}(n, x)\}$ it follows that $D_w \notin \mathfrak{D}(n, x)$. So

$$\begin{aligned}
U(m, n) &- \left(\bigcup \{U(D) \mid D \cap A \neq \emptyset \text{ and } D \in \mathfrak{D}(n, x)\}\right)^- \\
&\subseteq U(m, n) - \bigcup \{U(D) \mid D \cap A \neq \emptyset \text{ and } D \in \mathfrak{D}(n, x)\} \\
&= \bigcup \{U_w \mid w \in A_n \text{ and } n_w = m\} - \bigcup \{U(D) \mid D \cap A \neq \emptyset \text{ and } D \in \mathfrak{D}(n, x)\} \\
&\subseteq \bigcup \{\operatorname{St}(w, \mathcal{G}_{n_w}) \mid w \in D \cap A,\ D \in \mathfrak{D}_n - \mathfrak{D}(n, x) \text{ and } n_w = n_x = m\} \\
&\subseteq \bigcup \{\operatorname{St}(D, \mathcal{G}^2_{n_x}) \mid D \in \mathfrak{D}_n - \mathfrak{D}(n, x)\} = \operatorname{St}(E^n_x, \mathcal{G}^2_{n_x}).
\end{aligned}$$

Now recall that $\mathrm{St}(x, \mathcal{G}^2_{n_x}) \cap \mathrm{St}(E^n_x, \mathcal{G}^2_{n_x}) = \emptyset$ and that

$$\left(\bigcup\{U(D) \mid D \cap A \neq \emptyset \text{ and } D \in \mathfrak{D}(n, x)\}\right)^- \cap B = \emptyset.$$

So that

$$O_z = \mathrm{St}(x, \mathcal{G}^2_{n_x}) \cap \left(X - \left(\bigcup\{U(D) \mid D \cap A \neq \emptyset \text{ and } D \in \mathfrak{D}(n, x)\}\right)^-\right)$$

is an open neighborhood of z and $O_z \cap U(m, n) = \emptyset$, contradicting the assumption that $z \in U(m, n)^- \cap B$. Hence $U(m, n)^- \cap B = \emptyset$ for any $m \in N$. Finally for the fixed natural number n let

$$U_n = \bigcup_{m=1}^{\infty}\left[U(m, n) - \left(\bigcup_{i=1}^{n} W_i\right)^-\right] \quad \text{and} \quad W_n = \bigcup_{m=1}^{\infty}\left[W_m - \left(\bigcup_{i=1}^{n} U(i, n)\right)^-\right].$$

Since $W_i^- \cap A_n = \emptyset$ and $U(i, n)^- \cap B = \emptyset$ for each $i \in N$, it follows that $U_n \supseteq A_n$, $W_n \supseteq B$ and $U_n \cap W_n = \emptyset$. So in particular $U_n^- \cap B = \emptyset$.

Thus since n was an arbitrary member of N, it follows that we can construct a sequence $\{U_n \mid n \in N\}$ of open sets such that $U_n \supseteq A_n$ and $U_n^- \cap B = \emptyset$ for any $n \in N$.

By a dual argument we can construct a sequence $\{V_n \mid n \in N\}$ of open sets such that $V_n \supseteq B_n = B \cap (\bigcup\{D \mid D \in \mathfrak{D}_n\})$ and $V_n^- \cap A = \emptyset$ for each $n \in N$.

Thus if we let

$$U = \bigcup_{n=1}^{\infty}\left[U_n - \left(\bigcup_{i=1}^{n} V_i\right)^-\right] \quad \text{and} \quad V = \bigcup_{n=1}^{\infty}\left[V_n - \left(\bigcup_{i=1}^{n} U_i\right)^-\right],$$

then U and V are open sets such that $U \supseteq A$, $V \supseteq B$ and $U \cap V = \emptyset$. It follows that X is a normal topological space.

5. The Proof of Theorem 1.3. Having established that a T_2 space X with a refining sequence mod k is normal it is clear that X is a p-space. More precisely we have

5.1 Lemma. *If X is a T_2 space then the following are equivalent:*

(a) *X has a refining sequence* mod k.

(b) *X is a normal space with a p-sequence $\{\mathcal{H}_n \mid n \in N\}$ in βX such that $\bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{H}_n)^-_{\beta X} = \bigcap_{n=1}^{\infty} \mathrm{St}(C, \mathcal{H}_n) \subseteq X$ for each compact subset C of X.*

The proof of this lemma is straightforward and is omitted.

In §2 we defined Γ to be the first ordinal number whose cardinal number is the cardinal number of $\{P^{\omega}_x \mid x \in X\}$ and for each $\alpha \in \Gamma$ we assigned as element x_α in X such that $x_\alpha \notin P^{\omega}_{x_\beta}$ if $\alpha \neq \beta$. We also defined $P^k_\alpha = P^k_{x_\alpha}$ for each $k \in N$ and $P_\alpha = P^{\omega}_{x_\alpha}$. This terminology will be used below.

If $\mathfrak{D}$ is a discrete set in a space with a refining sequence $\{\mathcal{G}_n \mid n \in N\}$ mod k then by Lemma 2.9 there exists $n(\alpha, k) \in N$ and a finite subset $\mathfrak{D}(\alpha, k)$ of $\mathfrak{D}$ such that $\mathrm{St}\,(P^k_\alpha, \mathcal{G}_{n(\alpha,k)}) \cap \mathrm{St}\,(E^k_\alpha, \mathcal{G}_{n(\alpha,k)}) = \emptyset$ for each $\alpha \in \Gamma$ and each $k \in N$. Recall that $E^k_\alpha = \bigcup\{D \mid D \in \mathfrak{D} - \mathfrak{D}(\alpha, k)\}$. Let $\Gamma(k, j) = \{\alpha \in \Gamma \mid n(\alpha, k) = j\}$ and let $B(k, j, D) = \bigcup\{P^k_\alpha \cap D \mid \alpha \in \Gamma(k, j)\}$ for each $D \in \mathfrak{D}$. We then have:

5.2 **Lemma.** *If X is a T_2 space with a refining sequence* mod k *and* $\mathfrak{D}$ *is a closed discrete collection of subsets of X, then there exists a σ-discrete collection $\mathcal{O} = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} \mathcal{O}(k, j)$ of open subsets of X with the property that for each $B(k, j, D)$ (as defined above) there is $V(k, j, D) \in \mathcal{O}(k, j)$ such that $B(k, j, D) \subseteq V(k, j, D)$.*

Proof. Let $\Gamma(k, j)$ be defined as above and let α_0 be the first member of $\Gamma(k, j)$. Since $\mathfrak{D}(\alpha_0, k)$ is a finite collection of disjoint closed sets and X is normal by Theorem 4.2, there exist open sets $O(D)$ for each $D \in \mathfrak{D}(\alpha_0, k)$ such that $D \subseteq O(D)$ and $O(D) \cap O(D^*) = \emptyset$ whenever $D^* \neq D$, D^* and $D \in \mathfrak{D}(\alpha_0, k)$. Assume for $\beta > \alpha_0$, $\beta \in \Gamma(k, j)$ that $O(D)$ has been defined for each $D \in \mathfrak{D}(\alpha, k)$ where $\alpha \in \Gamma(k, j)$ and $\alpha < \beta$. If $\mathfrak{D}(\beta, k) - \bigcup\{D' \mid D' \in \mathfrak{D}(\alpha, k),\ \alpha \in \Gamma(k, j)$ and $\alpha < \beta\} = \emptyset$ then there is no problem. If this set is not empty let D be any member of it and let $O(D)$ be an open set such that

(a) $O(D) \supseteq D$,

(b) $O(D)^- \cap (\bigcup\{D^* \mid D^* \in D(\alpha, k),\ \alpha \in \Gamma(k, j) \text{ and } \alpha < \beta\}) = \emptyset$,

(c) $O(D) \cap O(D^*) = \emptyset$ if $D^* \neq D$ and $D^* \in \mathfrak{D}(\beta, k) - \bigcup\{D' \mid D' \in \Gamma(\alpha, k),$ $\alpha \in \Gamma(k, j)$ and $\alpha < \beta\}$.

By induction on the well ordering of $\Gamma(k, j)$ we define an open set $O(D)$ satisfying (a), (b), and (c) above for each $D \in \mathfrak{D}(\alpha, k)$ and each $\alpha \in \Gamma(k, j)$.

Let $\alpha \in \Gamma(k, j)$; let α_1 be the first member of $\Gamma(k, j)$ such that $\mathfrak{D}(\alpha, k) \cap \mathfrak{D}(\alpha_1, k) \neq \emptyset$ and let $\mathfrak{D}(\alpha, k) \cap \mathfrak{D}(\alpha_1, k) = \mathfrak{D}_1(\alpha, k)$. Let α_2 be the first member of $\Gamma(k, j)$ such that $[\mathfrak{D}(\alpha, k) - \mathfrak{D}(\alpha_1, k)] \cap \mathfrak{D}(\alpha_2, k) \neq \emptyset$ and let $\mathfrak{D}_2(\alpha, k) = [\mathfrak{D}(\alpha, k) - \mathfrak{D}(\alpha_1, k)] \cap \mathfrak{D}(\alpha_2, k)$. Continue this process until $\mathfrak{D}(\alpha, k) - \bigcup_{i=1}^{n} \mathfrak{D}_i(\alpha, k) = \emptyset$. Note that this process must terminate after a finite number of steps since $\mathfrak{D}(\alpha, k)$ is finite and note that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. For each i and each $D \in \mathfrak{D}_i(\alpha, k)$ let

$$\mathcal{O}(D, \alpha) = \left[\mathcal{O}(D) - \left(\bigcup\{\mathcal{O}(D') \mid D' \in \mathfrak{D}_m(\alpha, k) \text{ and } m > i\}\right)^-\right] \cap \mathrm{St}(P^k_\alpha \cap D, \mathcal{G}_j).$$

Note that if $m > i$ then $\alpha_m > \alpha_i$ and thus $O(D')^- \cap D \in \mathfrak{D}_m(\alpha, k) = \emptyset$ by condition (b) above. Thus

$$\left(\bigcup\{\mathcal{O}(D') \mid D' \in \mathfrak{D}_m(\alpha, k),\ m > i\}\right)^- \cap D$$

$$= \bigcup\{\mathcal{O}(D')^- \mid D' \in \mathfrak{D}_m(\alpha, k),\ m > i\} \cap D = \emptyset$$

and so $O(D, \alpha) \supseteq \mathrm{St}(P_\alpha^k \cap D, \mathcal{G}_j) \cap D$. For each $D \in \mathfrak{D}$, let $W(k, j, D) = \bigcup\{O(D, \alpha) \mid \alpha \in \Gamma(k, j)$ and $P_\alpha^k \cap D \neq \emptyset\}$ if there is some $\alpha \in \Gamma(k, j)$ such that $P_\alpha^k \cap D \neq \emptyset$ and let $W(k, j, D) = \emptyset$ otherwise. Note that $W(k, j, D) \subseteq O(D)$ since $O(D, \alpha) \subseteq O(D)$ for each $\alpha \in \Gamma(k, j)$.

We show that $B(k, j, D)^- \subseteq W(k, j, D)$: Let $z \in B(k, j, D)^-$; then $z \in \mathrm{St}(B(k, j, D), \mathcal{G}_j)$ for otherwise $\mathrm{St}(z, \mathcal{G}_j) \cap B(k, j, D) = \emptyset$. Hence it follows that $z \in \mathrm{St}(P_\alpha^k \cap D, \mathcal{G}_j)$ for some $\alpha \in \Gamma(k, j)$, since $B(k, j, D) = \bigcup\{P_\alpha^k \cap D \mid \alpha \in \Gamma(k, j)\}$. Clearly $D \in \mathfrak{D}(\alpha, k)$ and thus $z \in D \cap \mathrm{St}(P_\alpha^k \cap D, \mathcal{G}_j) \subseteq O(D, \alpha)$ since $B(k, j, D)^- \subseteq D$. Hence $B(k, j, D)^- \subseteq W(k, j, D)$. Next we show that $W(k, j, D) \cap W(k, j, D^*) = \emptyset$ for $D, D^* \in \mathfrak{D}$ and $D \neq D^*$. Suppose $z \in W(k, j, D) \cap W(k, j, D^*)$ for some $D, D^* \in \mathfrak{D}$ and $D \neq D^*$. So there exist α, and $\beta \in \Gamma(k, j)$ such that $z \in O(D, \alpha)$ and $z \in O(D^*, \beta)$. Let α^* and β^* be the first members of $\Gamma(k, j)$ such that $D \in \mathfrak{D}(\alpha^*, k)$ and $D^* \in \mathfrak{D}(\beta^*, k)$ respectively. Since $P_\alpha^k \cap D \neq \emptyset$ and $P_\beta^k \cap D^* \neq \emptyset$ we have that $D \in \mathfrak{D}(\alpha, k)$ and $D^* \in \mathfrak{D}(\beta, k)$. Thus α^* and β^* both exist. Recall that $W(k, j, D) \subseteq O(D)$, $W(k, j, D^*) \subseteq O(D^*)$ and by by condition (c) above $O(D) \cap O(D^*) = \emptyset$ if $\alpha^* = \beta^*$. Thus $\alpha^* < \beta^*$ or $\beta^* < \alpha^*$. It suffices to assume $\alpha^* < \beta^*$. Note that $\mathrm{St}(D^*, \mathcal{G}_j) \supseteq \mathrm{St}(P_\beta^k \cap D^*, \mathcal{G}_j) \supseteq O(D^*, \beta)$ and $\mathrm{St}(P_\alpha^k, \mathcal{G}_j) \supseteq \mathrm{St}(P_\alpha^k \cap D, \mathcal{G}_j) \supseteq O(D, \alpha)$. Thus it follows that $D^* \in \mathfrak{D}(\alpha, k)$ since $z \in O(D, \alpha) \cap O(D^*, \beta)$ and since $\mathrm{St}(P_\alpha^k, \mathcal{G}_j) \cap \mathrm{St}(D', \mathcal{G}_j) = \emptyset$ if $D' \notin \mathfrak{D}(\alpha, k)$. Since D and $D^* \in \mathfrak{D}(\alpha, k)$ and $\alpha^* < \beta^*$, then $D \in \mathfrak{D}_i(\alpha, k)$, $D^* \in \mathfrak{D}_m(\alpha, k)$ and $i < m$. So by definition $O(D, \alpha) \subseteq O(D) - O(D^*)$ and $O(D^*) \supseteq O(D^*, \beta)$ contradicting the assumption that $z \in O(D, \alpha) \cap O(D^*, \beta)$. Thus it follows that $W(k, j, D) \cap W(k, j, D^*) = \emptyset$ whenever $D \neq D^*$.

Now recall that $B(k, j, D)^- \subseteq D$ for each $D \in \mathfrak{D}$. Thus $\{B(k, j, D)^- \mid D \in \mathfrak{D}\}$ is a discrete collection and since X is normal there exists a discrete collection $\{V(k, j, D) \mid D \in \mathfrak{D}\}$ of open sets such that $B(k, j, D)^- \subseteq V(k, j, D) \subseteq W(k, j, D)$. Let $\mathcal{V}(k, j) = \{V(k, j, D) \mid D \in \mathfrak{D}\}$ and let $\mathcal{V} = \bigcup_{k=1}^\infty \bigcup_{j=1}^\infty \mathcal{V}(k, j)$. Clearly $\mathcal{V}$ is the desired collection.

Finally we prove Theorem 1.3.

Proof of Theorem 1.3. In [7] Filippov constructed a p-sequence $\{\mathcal{H}_n \mid n \in N\}$ for a paracompact p-space X satisfying the condition that $\bigcap_{n=1}^\infty \mathrm{St}(C, \mathcal{H}_n)^-_{\beta X} = \bigcap_{n=1}^\infty \mathrm{St}(C, \mathcal{H}_n) \subseteq X$ for each compact subset C of X. Thus by Lemma 5.1, a paracompact p-space has a refining sequence mod k.

Conversely assume X is a T_2 space with a refining sequence $\{\mathcal{G}_n \mid n \in N\}$ mod k. Then by Theorem 4.2 X is normal and hence by Lemma 5.1 is a p-space.

Let $\mathfrak{U}$ be any open cover of X. Since X is subparacompact by Theorem 3.2, there exists a σ-discrete closed refinement $\mathfrak{D} = \bigcup_{n=1}^\infty \mathfrak{D}_n$ for X. Thus for each $n \in N$ by Lemma 5.2 there exists a σ-discrete collection $\mathcal{V}_n = \bigcup_{k=1}^\infty \bigcup_{j=1}^\infty \mathcal{V}_n(k, j)$ of open sets such that for each $B_n(k, j, D)$ there exists $V_n(k, j, D) \in \mathcal{V}_n(k, j)$ with $B_n(k, j, D) \subseteq V_n(k, j, D)$ where $B_n(k, j, D) = \bigcup\{P_\alpha^k \cap D \mid D \in \mathfrak{D}_n\}$. Let $x \in X$; then there exists $k, n \in N$, $D \in \mathfrak{D}_n$ and $\alpha \in \Gamma$ such that $x \in D \cap P_\alpha^k$

hence $x \in B_n(k, j, D)$. Thus $\{B_n(k, j, D) \mid D \in \mathfrak{D}_n, n, k \text{ and } j \in N\}$ covers X.

For each n and each $D \in \mathfrak{D}_n$ let $U(D)$ be a fixed member of $\mathfrak{U}$ such that $D \subseteq U(D)$. Then let $U_n(k, j, D) = U(D) \cap V_n(k, j, D)$ for each k and $j \in N$. Clearly $U_n(k, j, D) \supseteq B_n(k, j, D)$. So

$$\mathfrak{U}^* = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} \{U_n(k, j, D) \mid D \in \mathfrak{D}_n\}$$

is an open refinement of $\mathfrak{U}$ and moreover $\{U_n(k, j, D) \mid D \in \mathfrak{D}_n\}$ is discrete since $\mathfrak{V}_n(k, j)$ is discrete. Thus it follows that $\mathfrak{U}^*$ is an open σ-discrete refinement of $\mathfrak{U}$. Hence X is paracompact and the proof is complete.

In conclusion it is observed that if X is a preimage of a metric space under a perfect map then it is fairly obvious that X has a refining sequence mod k. However, it is not at all obvious that if X is a space with a refining sequence $\{\mathcal{G}_n \mid n \in N\}$ mod k then it is a perfect preimage of a metric space. The problem is that $\{P_x^1 \mid x \in X\}$ need not partition the space X. Moreover if $\{\mathcal{G}_n \mid n \in N\}$ is refined in some way to make the resulting $\{P_x^1 \mid x \in X\}$ partition X, as can be done in subparacompact spaces, the new sequence need not be a refining sequence mod k.

BIBLIOGRAPHY

1. P. S. Aleksandrov, *On some results concerning topological spaces and their continuous mappings*, General Topology and its Relations to Modern Analysis and Algebra (Proc. Sympos., Prague, 1961), Academic Press, New York; Publ. House Czech. Acad. Sci. Prague, 1962, pp. 41–54. MR 26 #3003.

2. A. V. Arhangel'skiĭ, *On a class of spaces containing all metric and all locally bicompact spaces*, Dokl. Akad. Nauk SSSR 151 (1963), 751–754 = Soviet Math. Dokl. 4 (1963), 1051–1055. MR 27 #2959.

3. ———, *Bicompact sets and the topology of spaces*, Trudy Moskov Mat. Obšč. 13 (1965), 3–55 = Trans. Moscow Math. Soc. 1965, 1–62. MR 33 #3251.

4. ———, *On a class of spaces containing all metric spaces and all locally compact spaces*, Mat. Sb. 67 (109) (1965), 55–88; English transl., Amer. Math. Soc. Transl. (2) 92 (1970), 1–39. MR 32 #8299; 42 #3.

5. ———, *Mappings and spaces*, Uspehi Mat. Nauk 21 (1966), no. 4 (130), 133–184 = Russian Math. Surveys 21 (1966), no. 4, 115–162. MR 37 #3534.

6. D. K. Burke, *On subparacompact spaces*, Proc. Amer. Math. Soc. 23 (1969), 655–663. MR 40 #3508.

7. V. V. Filippov, *On the perfect image of a paracompact p-space*, Dokl. Akad. Nauk SSSR 176 (1967), 533–535 = Soviet Math. Dokl. 8 (1967), 1151–1153. MR 36 #5903.

8. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955. MR 16, 1136.

9. K. Morita, *Products of normal spaces with metric spaces*, Math. Ann. 154 (1964), 365–382. MR 29 #2773.

10. ———, *Some properties of M-spaces*, Proc. Japan Acad. 43 (1967), 869–872. MR 37 #3517.

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